Edge Detection from Spectral Phase Data

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Objectives

Image data are often collected as spectral samples for a number of applications. Reconstruction schemes typically rely on priors about edges in the image, hence the need for good edge detectors. However, the full set of data are rarely fully available in practice. The magnitude of Fourier data can be corrupt or lost to noise, leading to inaccurate edge detection and signal reconstructions. Thus, this study set out to explore:

- What edge information is unique to phase and magnitude data?
- What detection methods clearly separate phase and magnitude?
- Are phase-only edge detecting schemes robust to noisy or intermittent data?

Introduction

The N-th *Fourier partial sum* of a function f, $S_N f(x)$, is a finite harmonic approximation to f,

$$S_N f(x) = \sum_{k=-N}^{N} \hat{f}_k e^{ikx}$$

where the *Fourier coefficients* $\hat{f}_k$ are given by the Fourier transform of f

$$\hat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx.$$

Define the *jump function* of f, $[f](x)$, as the difference between the right and left hand limits of the function f at every point x,

$$[f](x) = f(x^+) - f(x^-).$$

Dirichlet devised a relationship between these limits and the Fourier partial sum approximation of f,

$$S_N f(x) = \sum_{k=-N}^{N} \hat{f}_k e^{ikx} \longrightarrow \frac{1}{2}\left(f(x^+) + f(x^-)\right).$$

The concentration factor edge detection method [1] introduces a *concentration factor* $\sigma(k)$ which modifies the partial sum to concentrate along the singular support of the underlying function so that the series converges to the jump function,

$$S_N^{\sigma}[f](x) = \sum_{k=-N}^{N} \sigma(k) \hat{f}_k e^{ikx} \longrightarrow [f](x).$$

Fourier Phase Data

The Fourier coefficients $\hat{f}_k$ are complex numbers; they can be written in *phasor form*

$$\hat{f}_k = A_k e^{i\phi_k}$$

where A_k is the *amplitude* and ϕ_k is the *phase*. If the amplitude is removed from the data, then only the phase ϕ_k is known.

If f is a 2π-periodic function on $[-\pi, \pi)$ with a single jump discontinuity at $x = \xi$, then integration by parts reveals a relationship between the magnitude and the phase of the Fourier coefficients,

$$\hat{f}_k \approx \frac{[f](\xi)}{2\pi i k} e^{-ik\xi} + \mathcal{O}\left(\frac{1}{k^2}\right) \approx |\hat{f}_k| e^{i\phi_k} + \mathcal{O}\left(\frac{1}{k^2}\right).$$

Concentration Factor Methods

Recently, a method for iteratively designing concentration factors was developed [2] allowing for greater flexibility and robustness to certain data constraints. Optimization is used to minimize

$$\|\mathbf{D}\sigma - [r](x)\|_2$$

where $\mathbf{D}$ is the DFT matrix, $[r](x)$ is the known jump function of the ramp $r(x)$, and σ is the solution vector. This solution scales to other functions since the Fourier coefficients of a piecewise smooth function f are scaled, shifted coefficients of the ramp,

$$\hat{f}_k \approx \frac{[f](\xi)}{2\pi i k} e^{-ik\xi} + \mathcal{O}\left(\frac{1}{k^2}\right) \approx [f](\xi) \hat{r}_k e^{-ik\xi} + \mathcal{O}\left(\frac{1}{k^2}\right).$$

Spectral Phase Data and Discontinuities

The locations and signs of jumps are entirely encoded within the phase of the spectral data. Thus, the analytical [1] and designed [2] concentration factors can be modified to approximate the jump function when only the phase information from the Fourier coefficients are given.

Computational Results

Figure 1: Jump Approximations from phase data

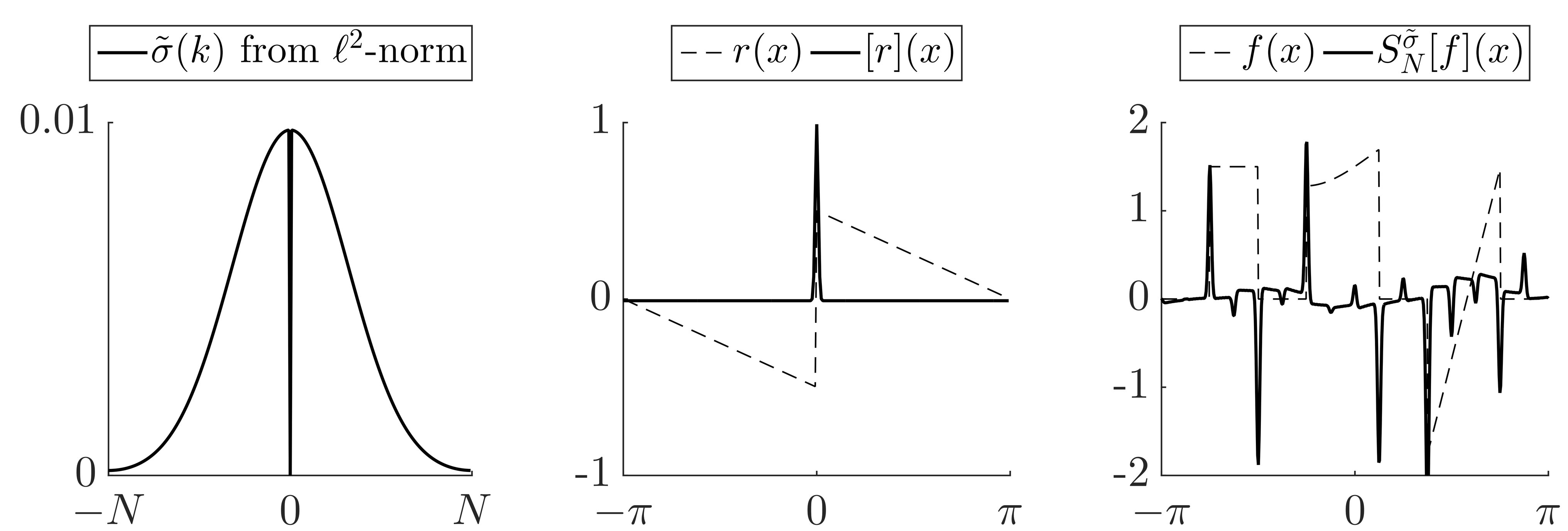
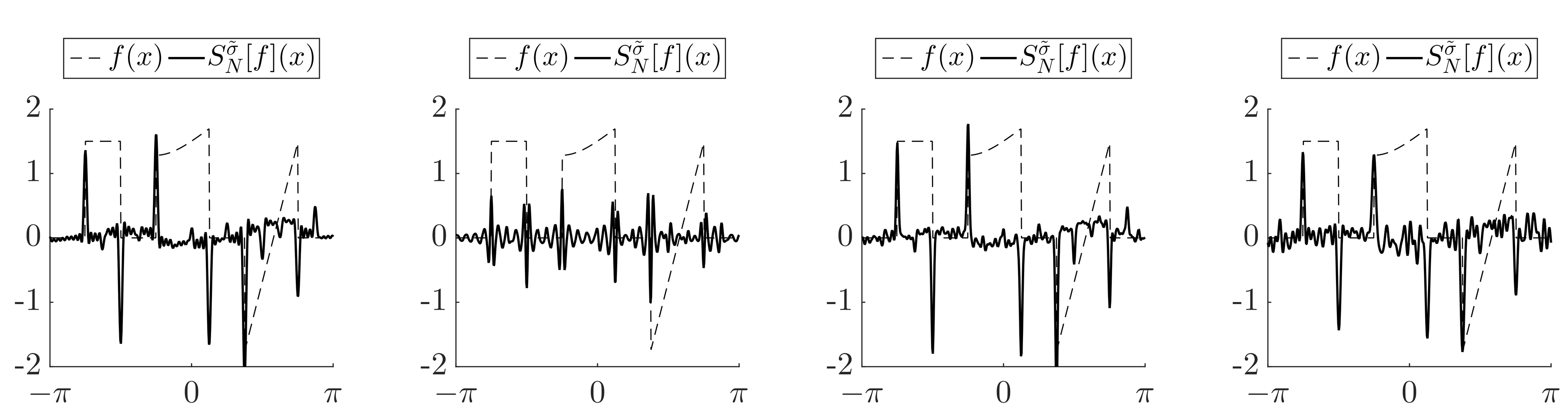


Figure 2: Jump Approximations from banded, noisy phase data



Conclusions and Future Work

A design approach allows for heavy flexibility in tailoring edge detection to the data given, including when magnitude information is removed. The phase information can be sensitive to the bands of data given, but hold up similar to full data under noise. The framework in [2] allows for combinations of multiple concentration factors. The effectiveness of a concentration factor can be evaluated through hypothesis testing [3]. Future work will be assessing whether combinations of concentration factors yields notable improvements to edge detection, especially with intermittent or noisy phase data.

References

[1] Gelb and Tadmor. “Detection of Edges in Spectral Data”. In: *ACHA* 7 (1999), pp. 101–135.

[2] Viswanathan, Gelb, and Cochran. “Iterative Design of Concentration Factors for Jump Detection”. In: *J Sci Comput* 51 (2012), pp. 631–649.

[3] Petersen, Gelb, and Eubank. “Hypothesis Testing for Fourier Based Edge Detection Methods”. In: *J Sci Comput* 51 (2012), pp. 608–630.

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